

The University of Chicago

QUADRATIC CREMONA
TRANSFORMATIONS IN FIVE-
DIMENSIONAL SPACE

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
DAVIS PAYNE RICHARDSON

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INTRODUCTION

The theory of plane Cremona transformations is complete. In the theory of space Cremona transformations the difficulties are legion and although much has been accomplished there is still a considerable part of the theory to be developed. One of the first difficulties and divergences from the plane theory is that the degree of the inverse transformation is not necessarily the same as that of the direct transformation. Some writers¹ believe that greater insight into the space theory will come from a study of Cremona transformations in higher dimensions and it is this idea which has prompted this particular study of a quadratic Cremona transformation in five-dimensional space.

In this paper are found the development of the equations of transformation, properties of the homoloidal family, correspondences between the two spaces, invariants when the spaces are superposed, and some special cases of the transformation when the parts of the F-system are not distinct or when the F-quadric is degenerate.

¹Hilda P. Hudson, Cremona Transformations, p. 395. Cambridge University Press, 1927.

EQUATIONS OF THE TRANSFORMATION

Let $(x) \equiv (x_0, x_1, x_2, x_3, x_4, x_5)$ be the homogeneous coordinates of a point in a five-dimensional x -space called S , and $(y) \equiv (y_0, y_1, y_2, y_3, y_4, y_5)$ be the homogeneous coordinates of a point in a five-dimensional y -space called S' supposed to exist quite independently of the S . Then the equations

$$(1) \quad y_0 : y_1 : y_2 : y_3 : y_4 : y_5 = x_0 x_1 : x_1 x_2 : x_2 x_3 : x_3 x_4 : x_4 x_5 : q,$$

where q is a homogeneous quadratic function in $x_0, \dots, x_4$, define a rational transformation from S to S' .

That this transformation is birational is evidenced by the fact that these equations can be solved for the x_i in terms of the y_i and this solution is unique. It is of the same form as the first transformation, viz:

$$(2) \quad x_0 : x_1 : x_2 : x_3 : x_4 : x_5 = y_0 y_5 : y_1 y_5 : y_2 y_5 : y_3 y_5 : y_4 y_5 : q',$$

where q' is the same quadratic function of $y_0, \dots, y_4$ that q is of $x_0, \dots, x_4$.

The five-dimensional (∞^5) system of transforming hyperquadrics,

$$(3) \quad \lambda_0 x_0 x_5 + \lambda_1 x_1 x_5 + \dots + \lambda_4 x_4 x_5 + \lambda_5 q = 0,$$

has a fundamental (or base) elements:

1. The simple point $O \equiv A_5(0, 0, 0, 0, 0, 1)$.

2. The quadratic three-dimensional variety, V_3^2 , intersection of $x_5 = 0$ and $q = 0$. This variety will be referred to in what follows as ω and the hyperplane $x_5 = 0$ will be called π .

Since the transformation is birational, (3) is a homoloidal system having the following properties:

1. The family is linear and is an ∞^5 system of varieties.
2. There is a single variable point of intersection of five general members.
3. The members must all be rational.
4. The postulation¹ of the system must be

$$\frac{1}{2} \cdot 6 \cdot 7 - 1 - 5 = 15$$

and its equivalence will be

$$2^5 - 1 = 31.$$

From (3) it is seen that properties 1 and 3 are satisfied.

As preliminary to showing that five general φ 's have a single variable point of intersection, which depends upon $\lambda_0, \dots, \lambda_5$, let us first consider the intersection of two quadric varieties which lie on a quadric variety of one higher dimension.

Lemma: Two quadric varieties V_{κ}^2 and v_{κ}^2 which lie on a quadric variety $V_{\kappa+1}^2$ intersect in a $V_{\kappa-1}^2$.

Proof: Any $V_{\kappa+1}^2$ lies in an $S_{\kappa+2}$ and any V_{κ}^2

¹The postulation and equivalence of a homoloidal family of degree n in r -space are given by

$P = \binom{n+r}{r} - 1 - r, \quad E = n^{\sim} - 1.$ Cf. Hudson, Cremona Transformations, p. 159. Also, see II, p. 6, of this paper.

lies in an S_{k+1} .¹ Since V_k^2 and v_k^2 are general quadric varieties lying on V_{k+1}^2 , the S_{k+1} in which V_k^2 lies and the $\bar{S}_{k+1}$ in which v_k^2 lies are general.

The intersection of S_{k+1} with V_{k+1}^2 , in S_{k+2} , is the V_k^2 of the hypothesis. As V_k^2 lies in S_{k+1} , its intersection with v_k^2 must lie in S_{k+1} ; also, since S_{k+1} meets V_{k+1}^2 only in V_k^2 , its intersection with v_k^2 must lie on V_k^2 . Therefore the intersection of V_k^2 and v_k^2 is the same as the intersection of S_{k+1} and v_k^2 . In S_{k+2} , a general S_{k+1} meets v_k^2 in a V_{k-1}^2 , which is the desired intersection.

Return now to the intersection of five general members, φ_5 , of the family (3). Two φ 's, say φ_1 and φ_2 , intersect in a V_3^4 , which, since it contains ω , breaks up into ω and a residual quadric variety ω_{12} . By the lemma, ω and ω_{12} intersect in a V_2^2 .

Also, φ_3 and φ_4 intersect in ω and ω_{34} , the latter being a quadric variety which meets ω in a second quadric surface v_2^2 , on ω in π .

Now V_2^2 and v_2^2 lie on ω , a V_3^2 , hence their intersection is a V_1^2 , which is a part of the intersection of ω_{12} and ω_{34} .

The total intersection of ω_{12} and ω_{34} is a V_1^4 , which breaks up into the V_1^2 just given and a v_1^2 which is not on ω .

¹Bertini, Introduzione alla Geometria Proiettiva degli Iperspazi, p. 228. 2nd edition, 1923. Casa Editrice Giuseppe Principato, Messina, Italy.

The fifth member of the family, φ_5 , intersects v_1^2 in four points; two of these are on ω since π meets v_1^2 in two points, one is 0, and the remaining point is variable. Thus all of the intersection of the five general φ_i except one variable point is included in 0 and ω .

Five general hyperquadrics in S intersect in thirty-two points. One can show just how the thirty-one points are accounted for in this way: ω in S_4 is determined by fourteen conditions; i.e., by fourteen points, v_2^2 in S_3 by nine, v_1^2 in S_2 by five, the two points of v_1^2 by two, and 0 by one condition. The total of these is thirty-one.

The point 0 and the quadric ω , through which all the members of the homoloidal family go, form the first fundamental system or first F-system. Equations (2) show that the homoloidal family in S' is the same kind as that in S , thus the second fundamental system will be a point 0' and a quadric variety ω' .

II

POSTULATION AND EQUIVALENCE

Postulation, P , for the base elements of a linear system is defined as the number of conditions these base elements present to the equations of the members of the family.

By definition, ω is a V_3^2 in an S_4 . Then there will be fifteen¹ homogeneous parameters in the equation of ω or fourteen non-homogeneous parameters. Therefore, to demand that a hyperquadric φ pass through ω imposes fourteen linearly independent conditions on the coefficients of φ . The point O is not on ω , hence for φ to pass through it imposes one more condition. Therefore

$$P \text{ for } (O, \omega) = 15.$$

Note: There are $\infty^{20} V_4^2$'s in S_5 . Imposing fifteen conditions gives $\infty^{20-15} = \infty^5 V_4^2$'s which are represented by (3).

Equivalence, E , for the base elements of a linear system in S_n , is defined as the number of points of intersection of n linearly independent members of the system that fall on the base elements.

As shown in section I, all but one of the thirty-two points of intersection are absorbed by O and ω . Therefore

$$E \text{ for } (O, \omega) = 31.$$

¹In the equation of a hypersurface V_{n-1}^r of S_n , there are $\binom{n+r}{r}$ homogeneous parameters. Bertini, loc. cit., p. 189.

III

THE JACOBIAN

The Jacobian of the family of φ 's is

$$J = -2x_5^4 q.$$

It consists of the hyperplane π counted four times and the hypercone q used in defining the transformation.

The Jacobian is the locus of double points of the φ 's. Since a φ which has a double point is a hyperquadric cone with this point as vertex, J is the locus of vertices of the cones of the family.

A φ may have more than one double point. If it has two double points, the line joining these lies wholly on φ and is a line of double points. Such a line of double points must lie on π or coincide with a generator of q .

If φ has three linearly independent double points the plane of these points lies entirely on the φ and is a plane of double points.

If φ has four linearly independent double points, the S_3 determined by these lies on φ and every point of it is double for φ . In this case φ degenerates into two hyperplanes.

If φ has five linearly independent double points, it is simply the hyperplane, determined by the five points, taken twice.

IV

CORRESPONDENCES BETWEEN S AND S'

We first consider the transforms or homologs of the F-systems and the frames of reference.

If the coordinates of O be substituted in (2) we have $0:0:0:0:0:1 = y_0 y_5 : y_1 y_5 : y_2 y_5 : y_3 y_5 : y_4 y_5 : q'$. Thus to the point O corresponds the hyperplane with equation $y_5 = 0$.

Given a point P(a), we define a_0, a_1, a_2, a_3, a_4 , or any set of numbers proportional to them, excluding $0, 0, 0, 0, 0$, to be the direction numbers of the line joining P(a) with $O \equiv (A_5)$. For, every point on this line has for its first five coordinates the numbers $a_0, \dots, a_4$, or numbers proportional to them, and conversely every point having numbers proportional to these as its first five coordinates lies on the line.

Now consider any point P(a) near O; this will be transformed into a definite point P' near π' . Take the point Q, on the line OP, which has coordinates $(a_0 \epsilon, a_1 \epsilon, a_2 \epsilon, a_3 \epsilon, a_4 \epsilon, \lambda + a_5 \epsilon)$; as $\epsilon \rightarrow 0$, $Q \rightarrow O$. Applying the transformation (1) we have as coordinates of the homolog $Q' [\epsilon a_0 (\lambda + a_5 \epsilon), \dots, \epsilon a_4 (\lambda + a_5 \epsilon), q(a_0 \epsilon, \dots, a_4 \epsilon)]$ where each term of q has ϵ^2 as a factor. On dividing by ϵ we have $Q' [a_0 (\lambda + a_5 \epsilon), \dots, a_4 (\lambda + a_5 \epsilon), \epsilon q(a)]$. Now let $\epsilon \rightarrow 0$, then $Q' \rightarrow \bar{Q}$ with coordinates proportional to $(a_0, a_1, a_2, a_3, a_4, 0)$.

Thus to O and the direction OP will correspond the point in π' which has as its first coordinates the

direction numbers of the line OP. If this direction happens to be along a generator of q , the corresponding point in S' will be on ω' and will have coordinates proportional to those of the point where this generator meets ω .

Consider P now a point on ω ; it will have coordinates $(a_0, a_1, a_2, a_3, a_4, 0)$ where $q(a) = 0$. On applying the transformation (1) we obtain $(0, 0, 0, 0, 0, 0)$ which has no meaning. Take a point $R(b)$ which is near P but not in π , (therefore not on ω). Then $b_5 \neq 0$. A point T on the line PR which is quite near P will have coordinates $T [1_0 + b_0\epsilon, a_1 + b_1\epsilon, \dots, a_4 + b_4\epsilon, b_5\epsilon]$ where ϵ is small. The transform of T by (1) is

$$T' [(1_0 + b_0\epsilon)^{-1}\epsilon, \dots, (a_4 + b_4\epsilon)b_5\epsilon, g(a + b\epsilon)].$$

For definiteness take $q = x_1^2 - x_2x_3 + x_0x_4$, which is projectively equivalent to all non-degenerate hypercones in S .

Using this particular q causes no loss of generality.

$$q(a + b\epsilon) = q(a) + \epsilon f(a, b) + \epsilon^2 q(b), \text{ where } f(a, b) = \sum a_i \frac{\partial q}{\partial b_i}, \quad q(a) = 0, \quad q(b) \neq 0.$$

On dividing the given coordinates of T' by $b_5\epsilon$ then letting $\epsilon \rightarrow 0$, i.e., $T \rightarrow P$, T' approaches $P' [a_0, a_1, a_2, a_3, a_4, \frac{f(a, b)}{b_5}]$. Now $f(a, b) \neq 0$ unless $R(b)$ lies in the hyperplane tangent to q at P . We shall restrict $R(b)$ from lying in this hyperplane and there are still ∞' choices for $R(b)$. The point P' is then on a generator of q' , the one whose direction numbers are proportional to the first five coordinates of the original point P on ω and whose actual position on this generator is determined by the direction of approach to P . And conversely, to a generator

of q' will correspond a point of ω' and the ω' directions from it not in π' .

Again, if $R(b)$ is on q , that is on the generator through P , for otherwise one could find a point on the line RP not on q , the point corresponding to P and the direction along the generator is the point, plus a certain direction, on ω' whose coordinates are proportional to those of P . For, in this case, $q(a + b\epsilon) \equiv 0$.

Suppose now that this point $R(b)$ is taken in π , not on ω . That is, consider a point on ω and a direction in π . Then $b_s = 0$, and T' becomes

$0, 0, 0, 0, 0, q(a - b)$ which is 0. As $\epsilon \rightarrow 0$, these coordinates remain unaltered, so a point on ω and a direction in π correspond to the F-point $0'$ and a direction from it along a generator of q' .

A point P in π but not on ω , will correspond to $0'$ plus a definite direction. Such a point has coordinates $(a_0, a_1, a_2, a_3, a_4, 0)$, where $q(a) \neq 0$, which upon applying (1) become $(0, 0, 0, 0, 0, 1)$.

The locus of points corresponding to the points of the F-system has been defined as the principal system or P-system. As has been shown, the loci corresponding to 0 and ω are π' , q' , and $0''$ which form the first P-system. Likewise π , q , and 0 form the second P-system.

Where parts of the P-system occur as factors in the equation of a transformed variety, they shall be dropped and the proper homolog will be the part given by the equation remaining.

We consider next the general hyperplane in S with equation

$$(4) \quad a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5 = 0.$$

When the transformation (2) is applied to this, the result is

$$(5) \quad a_0 y_0 y_5 + \dots + a_4 y_4 y_5 + a_5 q(y) = 0,$$

which is a hyperquadric.

Now if the hyperplane (4) goes through O , $a_5 = 0$ and (5) becomes

$$y_5(a_0 y_0 + \dots + a_4 y_4) = 0.$$

The factor $y_5 = 0$ is a part of the P -system and is to be dropped, leaving the expression in parentheses equal to zero, which is the equation of a general hyperplane in S' through O' , as the proper homolog of the hyperplane through O .

Since a hyperplane through O is transformed into a hyperplane through O' , the S_3 intersection of two hyperplanes through O corresponds to an S'_3 , the intersection of two hyperplanes through O' . In like manner the S_2 , or plane, intersection of three hyperplanes through O and the S_1 , or line, intersection of four hyperplanes through O , with the exception of the generators of q , will be transformed into a plane and line respectively.

The transformation (1) establishes a projectivity on the elements of each of these "stellas" or stars of linear spaces passing through O .

Equation (5) shows that a general hyperplane is transformed into a hyperquadric φ' . Since a general hyperplane does not go through O , its homolog φ' will meet π' in

ω' only. This hyperplane will intersect ω in a V_2^2 and since the points of ω correspond to the generators of q' , φ' will contain ∞^2 generators of q' . Also this hyperplane meets π in an S_3 , the points of which, excluding the V_2^2 on ω , have O' as homolog. Thus φ' has a simple four dimensional sheet through O' . The hyperplane meets every generator, h , of q once, hence φ' passes simply through

Consider two hyperplanes in S . The S_3 which is their intersection will correspond to the variable intersection of φ' and $\bar{\varphi}'$ which is a V_3^2 . ($V_4^2 \cdot V_4^2 = V_3^4 = \omega V_3^2$). Since S_3 will not in general pass through O , this variable intersection V_3^2 meets π' in a V_2^2 of ω' only. The S_3 will meet ω in a V_1^2 which means that the V_3^2 will contain ∞^1 generators of q' . This S_3 meets π in an S_2 , consequently the V_3^2 has a simple three dimensional sheet through O' . A part of this can be shown in a different way; since S_3 meets q in a V_2^2 the corresponding V_3^2 meets π' in a V_2^2 on ω' only.

Three general hyperplanes in S intersect in a plane, or S_2 , whose homolog will be the variable intersection of three φ' which is a V_2^2 . A general plane will not pass through O , so the V_2^2 will meet π' in a V_1^2 of ω' only. The plane meets ω in two points so V_2^2 will contain two generators of q' . The plane meets π in a line then V_2^2 will have a simple sheet, ordinary surface, through O' . Another way of arriving at a result given above is, since the plane meets q in a V_1^2 , the corresponding V_2^2 meets π' in a V_1^2 of ω' only.

A line, L , is the intersection of four general hyperplanes, hence its homolog will be the variable intersection of four φ' which is a V_1^2 . This line does not ordinarily pass through O , hence the V_1^2 meets π' in two points of ω' only. The line does not meet ω then V_1^2 meets q' in F -points only. It meets π in one point, hence V_1^2 passes simply through O' . Using a different view point, L meets q in two points, hence V_1^2 meets π' in two points of ω' only, as has been shown.

To the particular hyperplane π , corresponds the isolated point O' and its first neighborhood. It is understood that for points of π on ω we consider the directions around them which are in π .

To the hyperquadric cone q , considering only the directions from its vertex O which are along generators of q , will correspond ω' plus the directions from points of ω' not in π' . To a single point of q , not O , corresponds a single point of ω' with a particular direction from it and to a generator of q corresponds a single point of ω' with ∞' directions from it.

Any hyperquadric of the homoloidal family in S corresponds to a hyperplane in S' . A general hyperquadric f in S is transformed by (2) into a V_4^4 , in S' , which passes twice through ω' since f meets every generator of q twice. Since f meets ω in a V_2^4 the remaining intersection of V_4^4 with q' consists of ∞^2 generators of q' .

Each hyperplane of one frame of reference, except π and π' , corresponds to a hyperplane of the other frame of

reference. The equation $x_i = 0$ is transformed into $y_i = 0$, for $i = 0, 1, \dots, 4$.

A general variety $f^{(n)}(x) = 0$ is transformed by (2) into a variety $f^{(2n)}(y) = 0$ though the degree will be less than $2n$ if $f^{(n)}$ goes through any F-points.

As already given (p. 11) a hyperplane through O has as its homolog a hyperplane through O' . We shall need to know what other hyperplanes are transformed into hyperplanes.

Consider equation (4) with its transform (5), viz., $a_0 y_0 y_5 + \dots + a_4 y_4 y_5 + a_5 q(y) = 0$. If this equation is to represent a hyperplane, it must factor off a part of the P-system and leave a linear equation. If this hyperplane is not to go through O , $a_5 \neq 0$ then all but one of $a_0, \dots, a_4$ must vanish and q must be degenerate. Let $a_0 \neq 0$ and $a_1 = a_2 = a_3 = a_4 = 0$. Then $q' = y_0 g(y)$, where $g(y)$ is a linear function in $y_0, \dots, y_4$. Since y_0 is a part of q' , and is, consequently, a part of the P-system, it factors off leaving $a_0 y_5 + g(y) = 0$, which is the equation of a hyperplane. One may ask why could not $q' = y_5 g(y)$. It must be remembered that q can contain no term involving x_5 , nor can q' contain a term involving y_5 . We have established then the following

Theorem. A hyperplane in S which is transformed into a hyperplane in S' must have its equation in one of the forms

$$(6) \quad a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0$$

or

$$(7) \quad a_k x_k + a_5 x_5 = 0, \quad (k = 0, \dots, 4).$$

Corollary. A hyperplane not through O , which has equation of type (7), is transformed into a hyperplane only if q is degenerate.

An S_3 is given as the intersection of two hyperplanes. Since it is desired to have this transformed into an S_1 , the two hyperplanes must be transformed into hyperplanes. Hence the

Theorem. An S_3 which is to be transformed into an S_1 must be given as the intersection of two hyperplanes whose equations are of the form (6) or (7). If this S_3 does not go through O , then one equation must be of the form (7).

It is possible that planes and lines, which are to be transformed into planes and lines, be given as the partial intersection of hyperquadrics, since a non-degenerate hyperquadric can contain planes or lines. However, three hyperplanes can always be found which intersect in a given plane and four can be found which intersect in a line. Thus, planes and lines can always be expressed as intersections of hyperplanes.

If a plane is given as the intersection of three hyperplanes which are transformed into hyperplanes, their equations must be of the form (6) or (7). Not all three equations can be of type (7) unless two of them involve the same two coordinates. For, if three hyperplanes with equations of type (7), no two involving the same pair of coordinates, are to be transformed into hyperplanes, q' would have to be the product of three linear functions of $y_0, \dots, y_4$ which is impossible since q' is of the second degree.

However, it is not necessary that the hyperplanes defining the plane be transformed into hyperplanes since three hyperquadrics could intersect in a plane.

In like manner, if a line is given as the intersection of four hyperplanes which are themselves transformed into hyperplanes, their equations must be of type (6) and (7) where not more than two equations can be of type (7) unless two equations involve the same pair of coordinates.

A line through one point of ω corresponds to a line through one point of ω' , with the exception of generators of q and lines in π . This is readily proved by considering a point T on a line PQ where $P(a)$ is on ω and $Q(b)$ is any point not in π nor on q . Then $a_s = 0$, $q(a) = 0$, $b_s \neq 0$, and $q(b) \neq 0$. The coordinates of T can be written $(b_o + \lambda a_o, \dots, b_4 + \lambda a_4, b_5)$. The transform of T by (1) is $T' \left[(b_o + \lambda a_o)b_5, \dots, (b_4 + \lambda a_4)b_5, q(b) \right]$. It is obvious that as λ varies, T' varies on a line, not in π nor on q , but passing through one point of ω' .

SPACES SUPERPOSED IN A GENERAL WAY

1. The new frame of reference

Suppose the spaces S and S' be superposed in a general way; that is, so that the x - and y -frames of reference are entirely distinct. Choose a new frame of reference which has no particular relation to either of the given frames. If (Z) are the coordinates of a point referred to the new frame, the equations of transformation of the x and y coordinates respectively to this new frame are:

$$\rho x_i = \sum_j^{0..5} a_{ij} z_j, \quad (i = 0, \dots, 5)$$

(8)

$$\rho y_i = \sum_j^{0..5} b_{ij} z_j, \quad (i = 0, \dots, 5),$$

where the determinants $|a_{ij}|$ and $|b_{ij}|$ are both different from zero. Then the inverse transformations are given by:

$$\sigma z_i = \sum_j^{0..5} A_{ji} x_j, \quad (i = 0, \dots, 5),$$

(9)

$$\sigma z_i = \sum_j^{0..5} B_{ji} y_j, \quad (i = 0, \dots, 5),$$

where A_{ji} is the cofactor of a_{ij} in $|a_{ij}|$ and similarly for B_{ji} .

Let

$$\rho x_\alpha = \sum_j^{0..5} a_{\alpha j} z_j, \quad (\alpha = 0, \dots, 5)$$

(10)

$$\rho'_\alpha = \sum_j^{0..5} b_{\alpha j} z_j, \quad (\alpha = 0, \dots, 5).$$

Then equations (1) and (2) become

$$(11) \quad \begin{aligned} p'_0 : \text{----} : p'_5 &= p_0 p_5 : \text{----} : p_4 p_5 : q(p) \\ p_0 : \text{----} : p_5 &= p'_0 p'_5 : \text{----} : p'_4 p'_5 : q'(p'). \end{aligned}$$

From equations (11) we have

$$(12) \quad \begin{aligned} p_5 F_{01} &\equiv p_5 (p'_0 p_1 - p'_1 p_0) = 0 \\ p_5 F_{02} &\equiv p_5 (p'_0 p_2 - p'_2 p_0) = 0 \\ p_5 F_{03} &\equiv p_5 (p'_0 p_3 - p'_3 p_0) = 0 \\ p_5 F_{04} &\equiv p_5 (p'_0 p_4 - p'_4 p_0) = 0 \\ F_5 &\equiv p'_0 q(p) - p_0 p_5 p'_5 = 0 \end{aligned}$$

The first F-system is given by

$$O: \quad p_0 = p_1 = p_2 = p_3 = p_4 = 0, \quad \omega: \quad p_5 = 0, \quad q(p) = 0,$$

and the second F-system by

$$O': \quad p'_0 = p'_1 = p'_2 = p'_3 = p'_4 = 0, \quad \omega': \quad p'_5 = 0, \quad q'(p') = 0.$$

We turn now to the consideration of invariants when the spaces are superposed.

2. On Invariant points

If a point or locus of points is to be invariant under our transformation, equations (12) must be satisfied with $(x) = (y)$.

If an invariant point, I , lies on $p_5 = 0$, it is a general P-point which coincides with its homolog C' , and lies on $p'_0 = p'_1 = p'_2 = p'_3 = p'_4 = 0$, a situation which could not exist for a general superposition; or, it is an F-point of ω and coincides with a point on its corresponding P-line (generator of q'). In such a case we can choose p_1, p_2, p_3, p_4 , to pass through I . (A line is determined by four hyperplanes). Then I lies on q and q' , also on p'_1, p'_2, p'_3, p'_4 . Thus it is seen that if I is on p_5 , it is also on $F_{01}, F_{02}, F_{03}, F_{04}$, and F_5 .

If an invariant point I does not lie on p_3 , it must lie on all these F 's.

Therefore all invariant points lie on the intersections of $F_{01}, \dots, F_5$.

All of these varieties $F_{01}, \dots, F_5$ contain the S_3 : $p_0 = p'_0 = 0$, which is in general not an invariant space, nor will it in general contain any invariant planes, lines or points. If it is invariant, the x - and y -frames must have a particular relation, which is contrary to the hypothesis that the two spaces be superposed in a general way. If the F 's were general varieties of the dimensions and orders indicated, they would intersect in 48 points, but since they have this S_3 in common, the number of isolated invariant points will be less than 48: for,

$$V_4^2 \cdot V_4^2 = V_3^4 = V_3^3 S_3;$$

$$V_3^3 \cdot V_3^3 = V_1^9, \text{ which in general does not meet } S_3;$$

$V_4^3 \cdot V_1^9 = V_0^{27} = G_{17}$, a group of 27 points which is the greatest number of isolated invariant points. It may be that the F 's have other varieties in common in addition to the S_3 .

3. On Invariant Lines

If a line is to be invariant under the transformation, it must go through one F -point, either O or a point of ω . For a line through O corresponds to a line through O' , and a line through one point of ω corresponds to a line through one point of ω' .

Let OO' be an invariant line. Thence O lies on p'_5 and O' on p_5 . Take any four linearly independent

points in the S_3 , which is the intersection of p_5 and p_5' , together with O and O' to form the z -frame of reference.

Then OO' is given by:

$$z_5 = z_4 = z_3 = z_2 = 0.$$

We have then:

$$\begin{aligned} p_\alpha &= a_{\alpha 1} z_1 + \dots + a_{\alpha 5} z_5 \quad (\alpha = 0, \dots, 4) \\ p_5 &= z_0 \\ p_\alpha' &= b_{\alpha 0} z_0 + b_{\alpha 2} z_2 + \dots + b_{\alpha 5} z_5 \quad (\alpha = 0, \dots, 4) \\ p_5' &= z_1, \end{aligned} \quad (13)$$

where

$$O \text{ (looooo) and } O' \text{ (oloooo)}.$$

If any point of OO' is to be invariant

$$\text{then } x_5 = y_5 = x_4 = y_4 = x_3 = y_3 = x_2 = y_2 = 0,$$

$$\text{and } x_0 : x_1 = y_0 : y_1.$$

On substituting these values in (13) then

in (11) we have

$$b_{00} z_0 : - : - : - : b_{40} z_0 : z_1 = a_{01} z_1 z_0 : - : - : - : a_{41} z_1 z_0 : q(z)$$

from which

$$(14) \quad b_{00} : b_{10} : b_{20} : b_{30} : b_{40} : 1 = a_{01} : a_{11} : a_{21} : a_{31} : a_{41} : q(a)$$

Proof of (14)

$$\frac{b_{00} z_0}{b_{10} z_0} = \frac{a_{01} z_1 z_0}{a_{11} z_1 z_0}$$

$$b_{00} : b_{10} = a_{01} : a_{11}.$$

Similarly we get all the terms of the ratio except the last,

which is

$$\begin{aligned} \frac{b_{00} z_0}{z_1} &= \frac{a_{01} z_1 z_0}{q(a_{11} z_1 z_0)} \\ \frac{b_{00}}{1} &= \frac{a_{01} z_1^2}{q} = \frac{a_{01}}{q/z_1^2} \end{aligned}$$

Now q is homogeneous in z_0 and z_1 , and when divided by z_1^2 will contain three kinds of terms, viz., terms in z_0^2/z_1^2 , in z_0/z_1 , and constant terms. Since z_0/z_1 is constant $q(z)/z_1^2 = q(a_0, \dots, a_4)$.

These conditions do not decrease the order of the transformation as is seen from (14). They express the fact that the line lies on $F_{01}, F_{02}, F_{03}, F_{04}, F_{5-}$, but will not in general lie in the S_3 : $p_0 = p'_0 = 0$.

Suppose the invariant line L does not go through O , therefore not through O' . As shown on page 19, L meets ω in one point and ω' in one point.

Let L lie in π . Every line in π meets ω in two points unless it lies on ω . Now a point on ω and a direction in π correspond to O' and a direction, then the invariant L goes through O' and O hence it cannot lie in π as was supposed. Thus, isolated invariant lines cannot lie in π . The maximum number of invariant lines is given on page 24.

4. On Invariant Hyperplanes

If an invariant hyperplane goes through O , it goes also through O' .

Suppose the invariant hyperplane does not pass through O . Then every line in this hyperplane must meet ω and ω' . These must degenerate into two S_3 's each, one of each pair lying in the invariant hyperplane and the other in the P -hyperplane which is dropped in the transformation.

Proof: As shown on page 14, a hyperplane which is to be transformed into a hyperplane must have equation of the

form (6) or (7). Since this invariant hyperplane does not go through 0, it must be of the form (7). As shown there q' must break up into hyperplanes, as does q , and ω' must break up into two linear spaces, S_3 's, as does ω . One of the hyperplanes of q' is the P-hyperplane which therefore contains one S_3 of ω' . The other S_3 lies in the invariant hyperplane as is readily seen from the equation of the invariant hyperplane.

The maximum number of invariant hyperplanes can be found by finding the number of such which can satisfy equations (12). Since each F_{oi} ($i = 1, \dots, 4$) is a V_4^2 , the maximum number would be two invariant hyperplanes. If there are two such, $F_{o1} \equiv F_{o2} \equiv F_{o3} \equiv F_{o4}$ and $F_5 = HF_{o1}$, where $H = 0$ is an arbitrary hyperplane. This situation is impossible unless the p_i and p'_i are all identical, which cannot be if the chosen frame of reference is proper.

If there is one invariant hyperplane, W , $F_{o1}, \dots, F_{o4}$ factor into W times another hyperplane and $F_5 = W V_4^2$. In addition to W there are at least two isolated invariant points; for $S_4 \cdot S_4 \cdot S_4 \cdot S_4 \cdot V_4^2 = V_o^2$, which is two points. In case there is an invariant variety other than W , there need be no such two points.

5. On Invariant S_3 's

If there is an invariant S_3 through 0, it goes also through $0'$.

If there is an invariant S_3 not through 0, every line in it must go through one point of ω and one point of ω' . Therefore ω and ω' must each break up into two

S_3 's and the invariant S_3 will intersect one of each pair in a plane.

Proof: An S_3 which is transformed into an S_3 must be given as the intersection of two hyperplanes of type (7) or one of this type and one of type (8). As already shown, one of the hyperplanes of q' will be a P -hyperplane which will contain one of the S_3 's of ω' . The other S_3 will lie in one of the two hyperplanes giving the invariant S_3 . In a hyperplane two S_3 's meet in a plane. The two hyperplanes defining the invariant S_3 are not invariant.

The maximum number of invariant S_3 's is found by considering equations (12). Let us suppose that there is no invariant hyperplane. Then $F_{01}F_{02} = V_4^2 \cdot V_4^2 = V_3^4$. This variety of order four cannot consist of more than four linear spaces of dimension three. Assume the V_3^4 to be composed of four S_3 's. If $F_{01} \equiv F_{03}$ and $F_{02} \equiv F_{04}$, and if the four S_3 's are included in the intersection of F_{12} with F_{01} and F_{02} , then the four S_3 's are invariant; but such a situation is impossible if the z -frame of reference is proper. If however, no F 's are identital, the number of invariant S_3 's is at most two. It is possible to have one invariant hyperplane and one invariant S_3 at the same time.

6. On Invariant Planes

If an invariant plane goes through O it goes also through O' .

If an invariant plane does not go through O , every line in it must go through one point of ω and one point of ω' . Both ω and ω' degenerate into two S_3 's and one

line of an S_3 of each pair must lie in the invariant plane; i.e., the invariant plane must intersect one S_3 of ω and one S_3 of ω' each in a line.

The proof of this is quite the same as given in the two preceding cases and is omitted.

Since $V_4^2 \cdot V_4^2 \cdot V_4^2 = V_2^8$, the number of isolated invariant planes cannot exceed eight. Since the F 's go through the S_3 , $p_o = p'_o = o$, this maximum will not be reached.

$V_4^2 \cdot V_4^2 = V_3^4 = V_3^3 S_3$; $V_3^3 \cdot V_4^2 = V_2^6$. Thus it is seen that the maximum number of isolated invariant planes is six.

Since $V_4^2 \cdot V_4^2 \cdot V_4^2 \cdot V_4^2 = V_1^{16}$, the number of invariant lines cannot be greater than sixteen, but on account of the common S_3 this maximum is decreased to nine. For,

$$V_4^2 \cdot V_4^2 = V_3^4 = V_3^3 S_3, \quad V_3^3 \cdot V_3^3 = V_1^9.$$

7. On Invariant Varieties in General

Four general F 's intersect in a V_1^9 , a ninth degree curve, in addition to the common S_3 . If V_1^9 lies on F_5 , it may be invariant. Now the V_1^9 might degenerate into a V_1^k , $k < 9$, and $9 - k$ lines, in which case the V_1^k could be invariant. There can be no invariant curve of order greater than nine unless it be included in an invariant variety of dimension two or more.

If the $V_3^3 = F_{o1} \cdot F_{o2}$ and $V_3^3 = F_{o3} \cdot F_{o4}$ lie in the same hyperplane, $V_3^3 \cdot V_3^3 = V_2^9$, which could lie on F_5 , a V_4^3 , and hence be invariant.

It is possible that $V_3^3 \equiv V_3^3$ without any pair of the four F 's being identical, and that it lie on F_5 , in which case it could be invariant.

VI

SPECIAL CASES OF SUPERPOSITION

If S' be placed on S so that the vertices of the frames of reference coincide in some order, and so that the unit points of the two systems coincide, the situation will vary somewhat from that described in the general superposition.

The more nearly the two frames of reference coincide exactly, the greater the number of possible invariants.

Because A_5 has been taken as the vertex of q , it must be given special attention. Including the case where the frames coincide exactly, i.e., $A'_i \equiv A_i$, ($i = 0, \dots, 5$), there are 19 types of superposition.

As an illustration, we will consider the case where $A'_0 \equiv A_0$, $A'_1 \equiv A_2$, $A'_2 \equiv A_3$, $A'_3 \equiv A_4$, $A'_4 \equiv A_5$, $A'_5 \equiv A_4$.

The equations obtained from (1) and (2) corresponding to (12) are

$$\begin{aligned}
 (15) \quad & y_0 q - x_0 x_5 y_5 = 0 \\
 & y_1 q - x_1 x_5 y_5 = 0 \\
 & y_2 q - x_2 x_5 y_5 = 0 \\
 & y_3 q - x_3 x_5 y_5 = 0 \\
 & y_4 q - x_4 x_5 y_5 = 0 \\
 & y_5 q - x_5^2 y_5 = 0
 \end{aligned}$$

When we substitute x_0 for y_0 , x_2 for y_1 , etc., these equations become

$$\begin{aligned}
 (16) \quad & x_0(q - x_5x_4) = 0 \\
 & x_2q - x_1x_3x_4 = 0 \\
 & x_3q - x_2x_5x_4 = 0 \\
 & x_1q - x_3x_5x_4 = 0 \\
 & x_5(q - x_4^2) = 0 \\
 & x_4(q - x_5^2) = 0
 \end{aligned}$$

The invariant loci are the loci of points whose coordinates satisfy these equations and hence lie on the intersection of $q - x_5x_4 = 0$, $x_2 - x_1 = 0$, $x_3 - x_2 = 0$, $x_5 - x_4 = 0$, which is a V_1^2 , or an S_2 lying on $q - x_5x_4 = 0$.

VII

SPECIAL CASES OF THE TRANSFORMATION

1. The Qudaric ω Degenerates into Two S_3 's

This special case has already been introduced in the discussion on certain invariant spaces. (pages 21 and 22) There are a few special properties which should be mentioned here.

Let the two S_3 's of ω be

$$x_5 = 0, \quad x_0 = 0; \quad x_5 = 0, \quad x_1 = 0.$$

Then $q = x_0 x_1 = 0$.

The equations of transformation become

$$(17) \quad y_0 : --- : y_4 : y_5 = x_0 x_5 : --- : x_4 x_5 : x_0 x_1,$$

$$(18) \quad x_0 : --- : x_4 : x_5 = y_0 y_5 : --- : y_4 y_5 : y_0 y_1,$$

from which it is seen that the inverse transformation is specialized in the same way.

$$J = -2 x_5^4 q = -2 x_5^4 x_0 x_1.$$

The plane of double points of ω , which has equations $x_5 = x_0 = x_1 = 0$, is a plane of double points of every φ of the homoloidal family

$$x_5 (\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4) + \lambda_5 x_0 x_1 = 0.$$

It can be shown that the family of S_3 's containing the double plane of ω is transformed into a family of S_3 's containing the double plane of ω' .

The general S_3 through the double plane is given by:

$$\begin{cases} a_0 x_0 + a_1 x_1 = 0 \\ b_0 x_0 + b_1 x_1 = 0 \end{cases}$$

From these equations it is seen that there are ∞^2 such S_3 's.

The number of S_3 's through a plane in S_5 is ∞^2 ¹.

Thus the equations give all S_3 's through the plane. On applying the transformations (17) and (18) it is evident that the result is the ∞^2 S_3 's through the double plane of ω' .

It is possible that ω might be two coincident S_3 's, say $x_5 = x_0 = 0$. Then $q = x_0^2$, and the equations of transformation will have this change. In this case every point of ω is double and this S_3 lies on and is a double S_3 for every φ of the homoloidal family. This S_3 is the intersection of hyperplanes tangent at any two points of the φ 's.

In this case it can readily be shown that all hyperplanes through this S_3 are transformed into all hyperplanes through the double S_3 which is ω' , thus establishing a projectivity.

2. The F-point 0 lies on the F-hyperplane π .

If 0 is on π , it must then be on ω since π cuts any φ in ω and at no other point if φ is proper.

Let $q \equiv x_1^2 - x_2x_3 + x_0x_4$, the particular one used on page 9, then ω will be

$$\begin{cases} x_5 = 0 \\ x_1^2 - x_2x_3 + x_0x_4 = 0 \end{cases}$$

¹Bertini, loc. cit., p. 39.

The point O cannot be at the vertex A_5 , as heretofore, so we shall let it approach A_0 (looooo), which is on ω , along the line A_5A_0 . That is, we take O with coordinates $(loooo\epsilon)$ and let $\epsilon \rightarrow 0$. The equations of the line A_5A_0 are $x_1 = x_2 = x_3 = x_4 = 0$. Since O does not lie at A_5 , we must replace x_5x_0 by $x_5(x_5 - \epsilon x_0)$ in equations (1) and when $\epsilon \rightarrow 0$, these equations become

$$(19) \quad y_0 : --- : y_5 = x_5^2 : x_1x_5 : x_2x_5 : x_3x_5 : x_4x_5 : (x_1^2 - x_2x_3 + x_0x_4)$$

The homoloidal family is given by

$$(20) \quad x_5 (\lambda_0 x_5 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4) + \lambda_5 (x_1^2 - x_2x_3 + x_0x_4) = 0$$

If, merely for the sake of symmetry, we re-number the

φ 's and instead of (19) we take

$$(21) \quad y_0 : --- : y_5 = (x_1^2 - x_2x_3 + x_0x_4) : x_5^2 : x_1x_5 : x_2x_5 : x_3x_5 : x_4x_5,$$

then the inverse transformation is

$$(22) \quad x_0 : --- : x_5 = (y_0y_1 - y_2^2 + y_3y_4) : y_2y_1 : y_3y_5 : y_4y_5 : y_5^2 : y_1y_5.$$

Equations (22) show that O' lies on ω'

Every φ of the family (20) goes through the point (looooo) and has there the same tangent hyperplane, viz.,

$$x_4 = 0.$$

In the Jacobian of this family, viz.,

$$J = 2x_5^5x_4,$$

the P-hypercone is replaced by the two hyperplanes $x_4 = 0$, $x_5 = 0$.

If a general variety, $f^{(22)}$, passes once through O , each term has one of $x_1, \dots, x_5$ as a factor and π' is dropped from the equation of the homolog f' . If, at O , f has $x_4 = 0$ as tangent hyperplane, then π'^2 is dropped from f' . If f goes through ω , therefore through O , y_5y_4

is dropped from f' .

Let $f'^{(h)}$ have an i -tuple point at $O(100000)$.

Then since all derivatives of order $i - 1$ of f vanish for O , the equation of f can be written

$$f \equiv u_i x_0^{n-i'} + u_{i+1} x_0^{n-i-1} + \dots + u_n = 0$$

where u_k is a homogeneous function of degree k in

$x_1, \dots, x_5$. The tangent hypercone at O is $u_i = 0$.

It is possible for this tangent hypercone to contain the hyperplane $x_4 = 0$ any number of times from 0 to i .

If it contains this hyperplane i times the equation $u_i = 0$ reduces to $x_4^i = 0$ and the tangent hypercone is simply the hyperplane counted i times.

Suppose the tangent hypercone contains x_4 k times as a factor, where $0 < k \leq i$. Then f' , the transform of f , will contain π' as a factor μ times where $\mu \leq k + i$. If x_4 is tangent to k separate sheets of f , then $\mu = k + i$. In such a case π' is a factor i times because O is an i -tuple point and k times because x_4 is tangent to k separate sheets there. If some sheets are cuspidal in nature or have higher than simple contact with the hyperplane $x_4 = 0$, then $k > i$.

Let us consider the transformation of the neighborhood of the point O . Let $P(\epsilon, \eta_1\epsilon, \eta_2\epsilon, \eta_3\epsilon, \eta_4\epsilon, \eta_5\epsilon)$ be a point near O which approaches O along any path as $\epsilon \rightarrow 0$. Applying (21) to this we have as the corresponding point $P'[(\eta_1^2\epsilon - \eta_1\eta_2\epsilon + \eta_4), \eta_5^2\epsilon, \eta_1\eta_3\epsilon, \eta_2\eta_5\epsilon, \eta_3\eta_5\epsilon, \eta_4\eta_5\epsilon]$ and $P' \rightarrow O'(100000)$ unless η_4 is small of the same order as ϵ . If so, let $\eta_4 = \eta'_4\epsilon$, where η'_4 is finite,

then $P \rightarrow O$ along a path which is approximately a conic tangent to the hyperplane $x_4 = 0$. For this value of η_4 as $\epsilon \rightarrow 0$, $P' \left[(\eta_1^2 - \eta_2 \eta_3 + \eta_4'), \eta_5^2, \eta_1 \eta_5, \eta_2 \eta_5, \eta_3 \eta_5, \eta_4 \eta_5 \right]$ approaches a definite point in π' distinct from O' .

Thus to general points near O corresponds the single point O' . Points in the neighborhood of O lying near the fixed tangent hyperplane correspond to definite points in π' distinct from O' .

3. Cases 1 and 2 combined.

Let the two S_3 's of ω be distinct and O be at a general point of one of them. The equations of the transformation can be

$$(23) \quad y_0 : \dots : y_5 = x_0 x_1 : x_2^2 : x_1 x_5 : x_2 x_5 : x_3 x_5 : x_4 x_5$$

$$(24) \quad x_0 : \dots : x_5 = y_0 y_1 : y_2^2 : y_2 y_3 : y_2 y_4 : y_2 y_5 : y_2 y_6$$

$$J = 2 x_5^5 x_1$$

Now all the φ 's of the family

$$(25) \quad x_0 x_1 + x_5 (\lambda_1 x_5 + \lambda_2 x_1 + \lambda_3 x_2 + \lambda_4 x_3 + \lambda_5 x_4) = 0$$

contain the double plane of ω but are not necessarily tangent to the fixed hyperplane $x_1 = 0$.

VITA

Davis Payne Richardson was born in Ben Franklin, Texas, October 10, 1902. His common school education was begun at Cane Hill, Arkansas, and completed at Fayetteville, Arkansas. He attended the University of Arkansas, receiving the B.A. degree in 1922. Since that time he had attended Harvard University one year, taught at Georgia School of Technology two years, taught at the University of Arkansas two years, and attended the University of Chicago two years in addition to three Summer quarters.

During his stay at the University of Chicago he has studied under Professors Bliss, Dickson, Lunn, MacMillan, Lane, Logsdon, Bartky, and Barnard. To all of these he expresses appreciation, in particular to Professor Logsdon under whose direction this thesis was written.

